

Chirp Sources

Hyperbolic Frequency Modulation

The form of a an Hyperbolically Frequency Modulated (HFM) signal is given by

$$s(t) = w(t) \sin(a \ln(1-kt)) \quad 0 \leq t \leq T$$

where $w(t)$ is a windowing function, and the chirp sweep is determined by the argument of the sine. In this case

$$k = (f_2 - f_1) / f_2 T$$

$$a = -2\pi f_1 / k = \frac{2\pi f_1 f_2}{(f_2 - f_1)} T$$

and

f_1 is the initial frequency, f_2 the final frequency, and T is the pulse duration. The phase of the sine term is

$$\phi(t) = \frac{f_1}{k} \ln(1-kt)$$

corresponding to an instantaneous frequency of

$$f_i = \frac{d\phi}{dt} = \frac{f_1}{(1-kt)}$$

After substituting for k and rearranging yields

$$\frac{1}{f_i} = \frac{1}{f_1} \left(1 - \frac{t}{T}\right) + \frac{1}{f_2} \frac{t}{T}$$

The frequency sweeps linearly between the reciprocal of the lower and upper frequency; hence the name hyperbolic frequency modulation. The spectral level of the HFM source decays across the pass band as the frequency increases and is given approximately by

$$A = \frac{\sqrt{f_1 f_2}}{f} \sqrt{\frac{T}{\Delta f}}$$

The spectrum can be found directly by Fourier transform or approximately by applying stationary phase to $\exp(i\phi(t) - \omega t)$ or equivalently by replacing the limits of the FT by infinity.

The windowing function $w(t)$ is approximately a boxcar and its transform is approximately a sinc function with bandwidth $2/T \sim 1$ Hz, thus the frequency content of the signal is dominated by the spectrum of HFM sweep. The windowing function acts to slightly smooth the spectrum of the signal in the frequency domain.

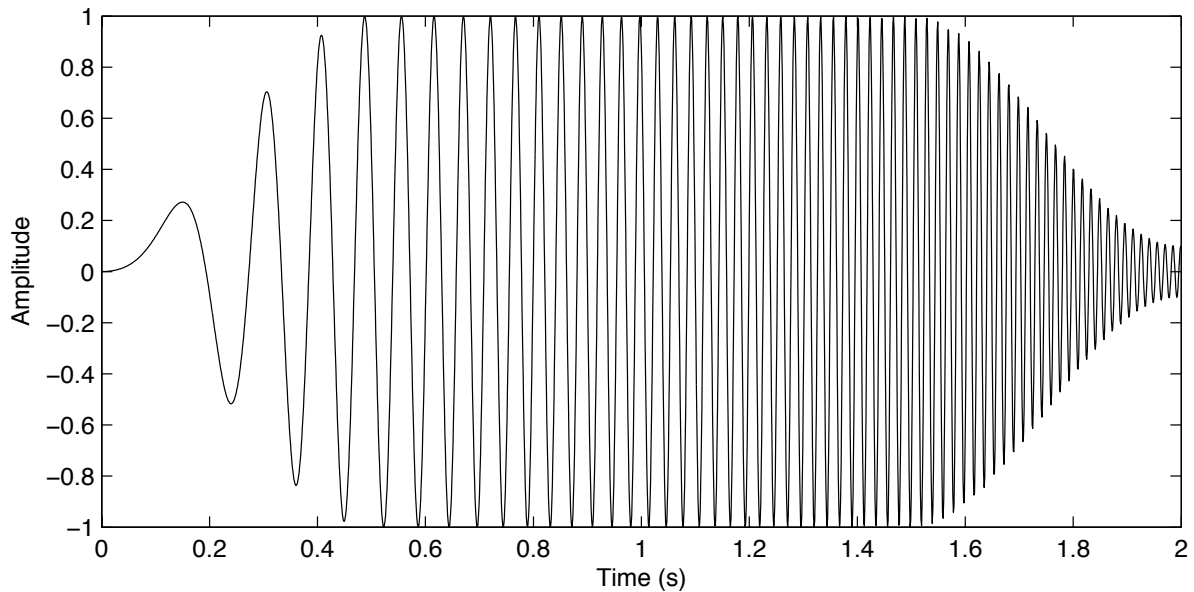
In the 1992 ARSRP reconnaissance experiment, a HFM source with a duration of 2 s. was used. The pulse used an upsweep or downsweep over the frequency range 210-280 Hz. The demodulation frequency, f_o , was set at 250 Hz (see below). The windowing function for the pulse was a Tukey window given by

$$w(t) = \begin{cases} p + (1-p)\sin^2\left(\frac{\pi}{2} \frac{t}{T_w}\right) & \text{for } 0 \leq t \leq T_w \\ 1 & \text{for } T_w \leq t \leq T - T_w \\ p + (1-p)\sin^2\left(\frac{\pi}{2} \frac{T - t}{T_w}\right) & \text{for } T - T_w \leq t \leq T \end{cases}$$

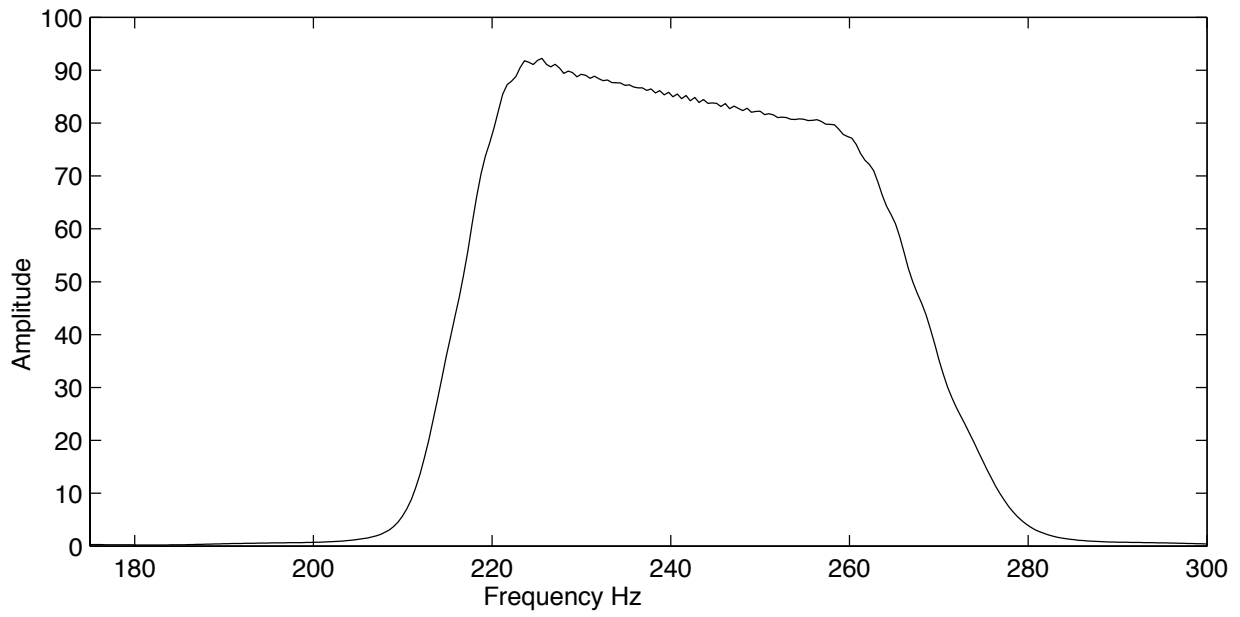
For the ARSRP reconnaissance experiment, the upsweep HFM had parameters

$$f_1 = 210 \text{ Hz}, f_2 = 280 \text{ Hz},$$

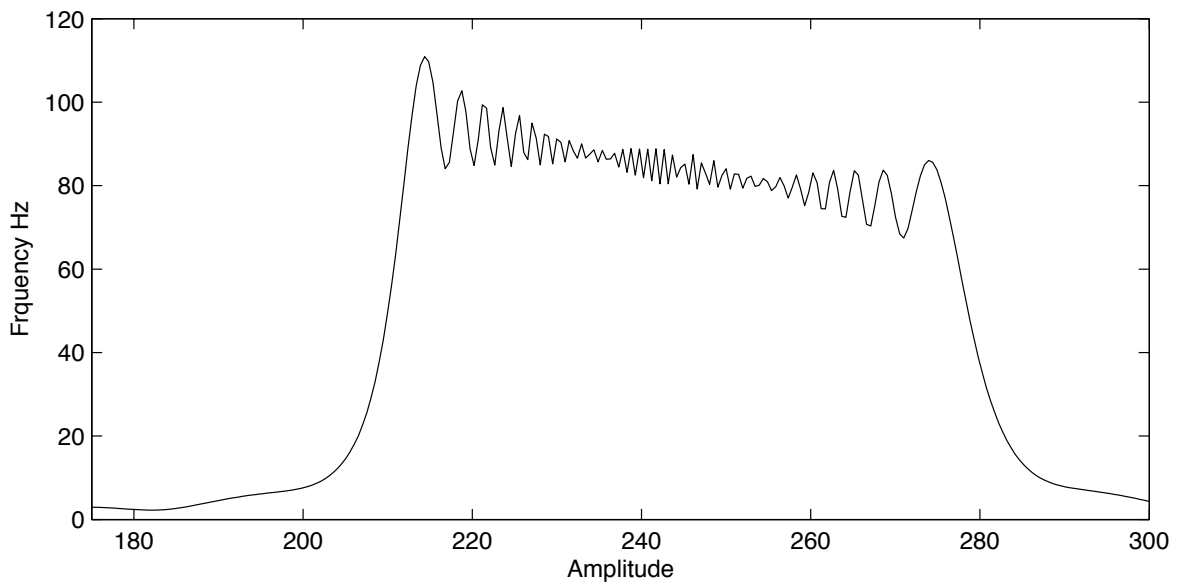
$$T = 2 \text{ s}, T_w = 1/8T = 1/4 \text{ s}, \text{ \& } p = 0.1.$$



SRP 210-280 Hz HFM source demodulated using $f_0 = 210$ Hz, effectively creating a source from 0–70 Hz.



Spectrum of HFM source including the window taper $w(t)$.



Spectrum of HFM source without the window taper $w(t)$

LFM Source

A Linear Frequency Modulated (LFM) source is can be represented as

$$s(t) = w(t) \sin\left(f_1 t + \frac{1}{2T}(f_2 - f_1)t^2\right) \quad 0 \leq t \leq T$$

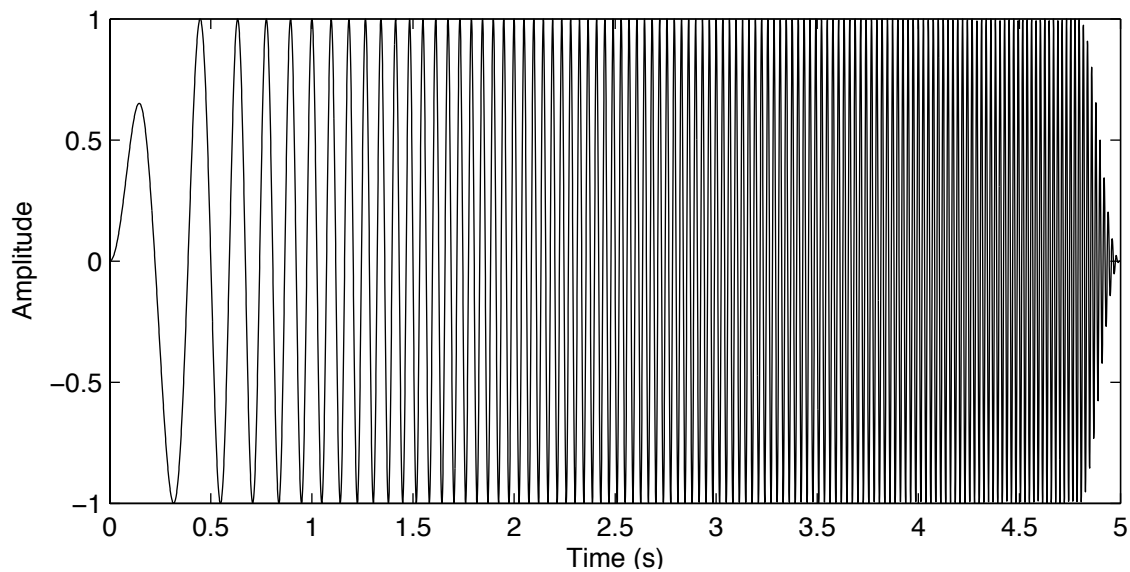
The instantaneous frequency, derivative of phase, is

$$f_i(t) = \left(1 - \frac{t}{T}\right) f_1 + \frac{t}{T} f_2$$

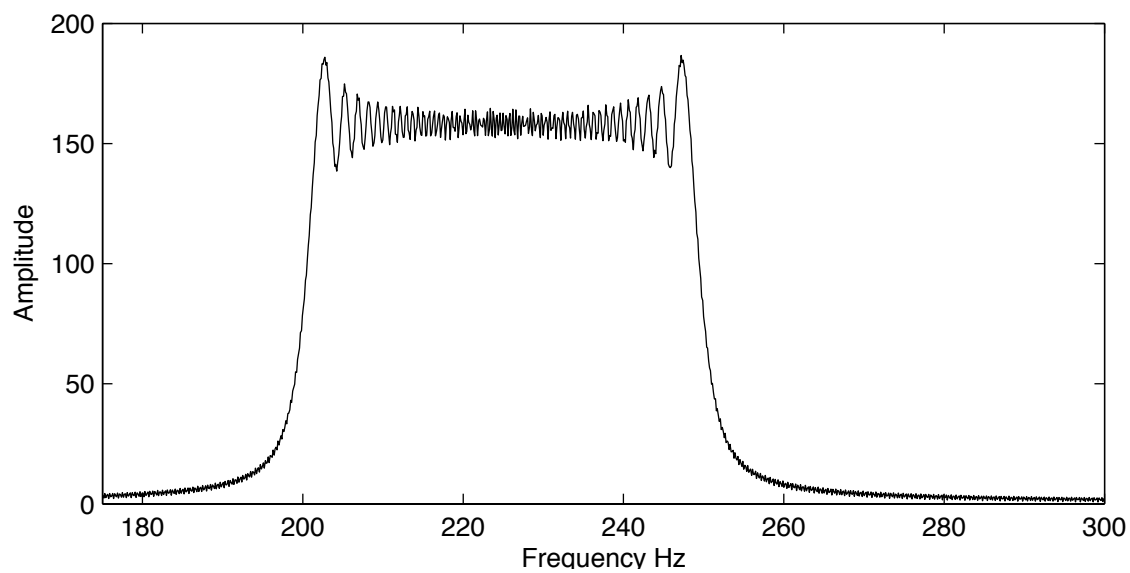
The level of the frequency spectrum, found by stationary phase, is approximately

$$\left(\frac{df_i}{dt}\right)^{-1/2} = \sqrt{\frac{T}{\Delta f}} = 0.302$$

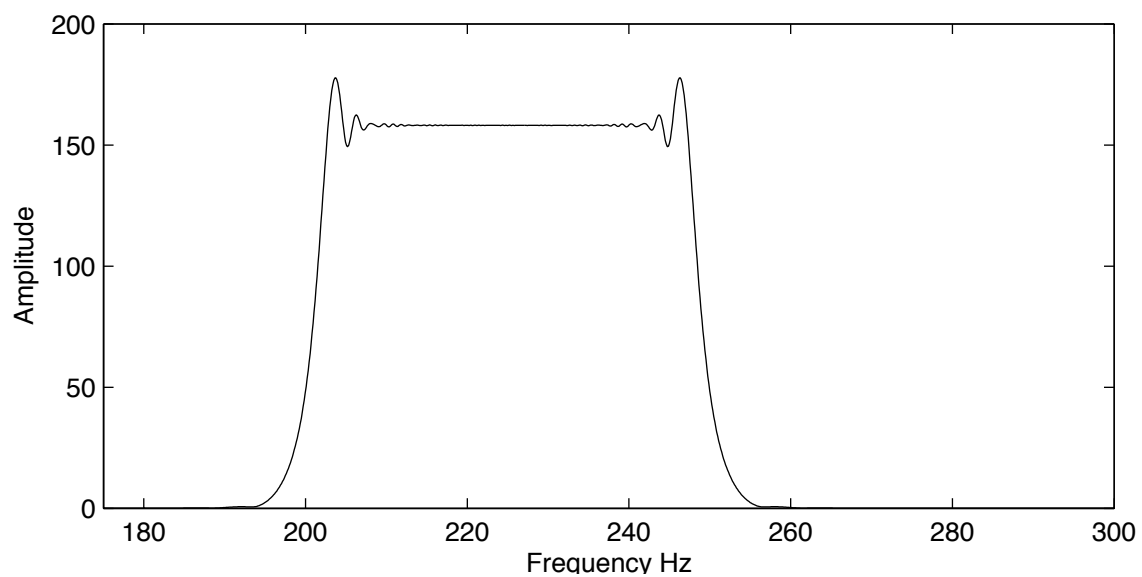
For the 1993 SRP main experiment, a source with $f_1 = 200$, $f_2 = 255$, and $T = 5$ s was used. The windowing function $w(t)$ was again a Tukey window, as above, but with $p = 0$; and $T_w = 0.04T = 0.2$ s.



SRP 200-255 Hz LFM source, demodulated using 200 Hz, effectively creating a 0-55 Hz LFM source



Spectrum of LFM pulse without window function $w(t)$.

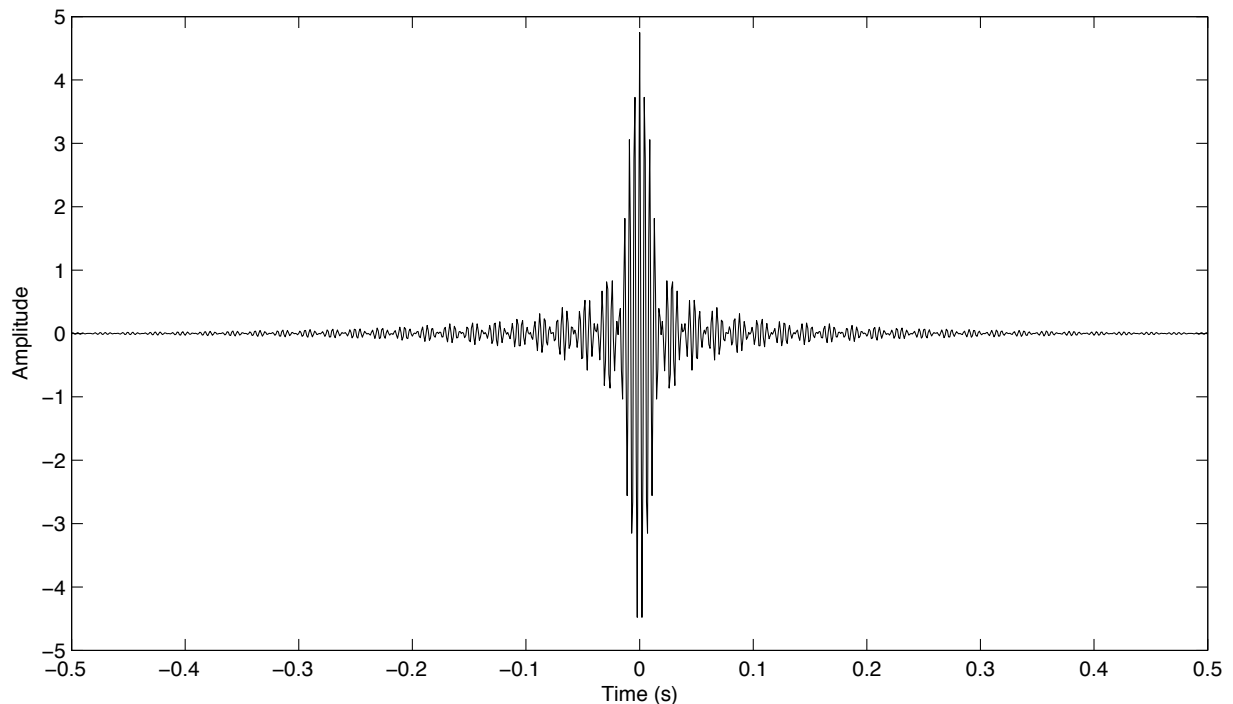


Spectrum of LFM pulse with windowing $w(t)$.

Matched Filtering

Matched filtering, is equivalent to cross-correlating the data with the original chirp signal. After cross-correlation the effective source signal is the autocorrelation of the original signal, which is equivalent to the inverse fourier transform of the zero-phase power spectrum.

For the LFM signal in the previous example, the cross-correlation yields a relatively compressed pulse, that oscillates relatively rapidly at approximately at the mid-frequency of the sweep, 227 Hz.

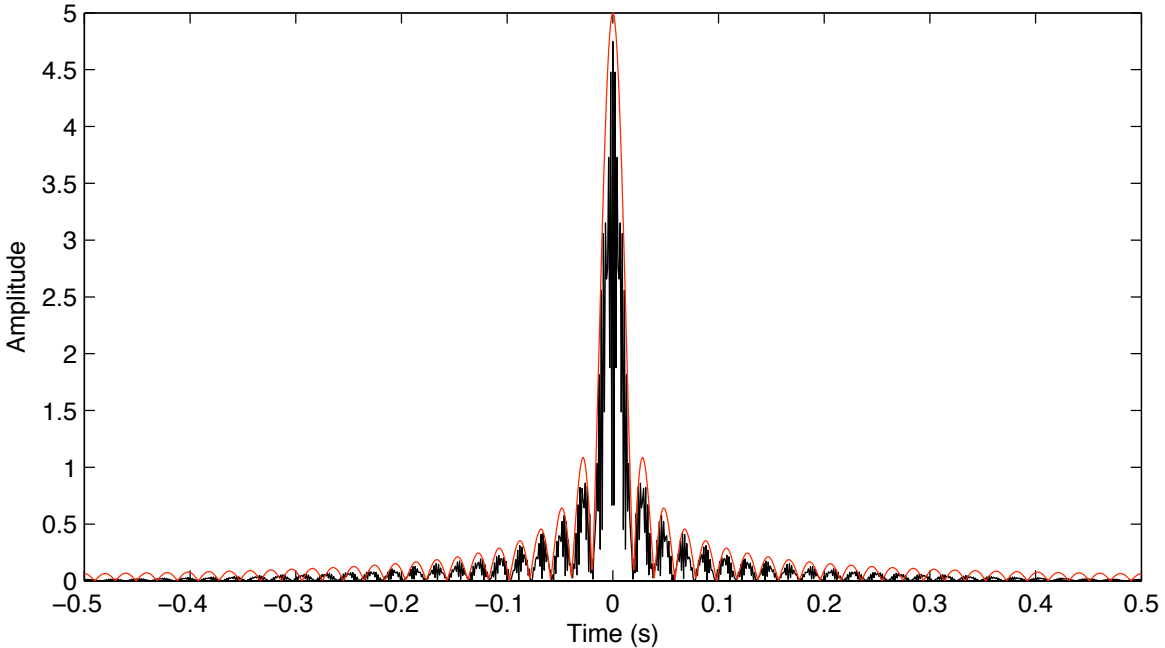


Cross-correlation of 200-255 Hz LFM pulse.

The MF pulse can be approximated by a modulated sinc function, since its power spectrum is approximately uniform. Thus

$$MF(t) \approx T \operatorname{sinc}(\pi \Delta f t) e^{i\pi(f_1 + f_2) t}$$

where $\Delta f = f_2 - f_1$, is the bandwidth of the chirp. The time resolution of the MF signal, depends on the decay of the maximum amplitude, which is defined by the sinc function in the approximate response. Thus the time resolution of the chirp signal is inversely proportional to chirp bandwidth. The following figure compares the absolute amplitude of the chirp signal to the sinc function envelope. With the window pulse, the best fit was obtained with $\Delta f = 51$ Hz.



Comparison of absolute amplitude of MF signal with sinc function for $\Delta f = 51$ Hz.

Complex Representation of Modulated Signals

A modulated narrowband signal can be represented as

$$s(t) = \text{Re}[\tilde{s}(t)e^{i2\pi f_c t}]$$

where f_c is the carrier frequency and the baseband signal, is complex consisting of real and imaginary signals, or, in engineering parlance, in phase and quadrature signals

$$\tilde{s}(t) = s_I(t) + is_Q(t)$$

If the spectrum of the baseband signal is non-zero only for $f > 0$ then the imaginary part of the signal is the Hilbert transform $s^\dagger(t)$, of the real signal.

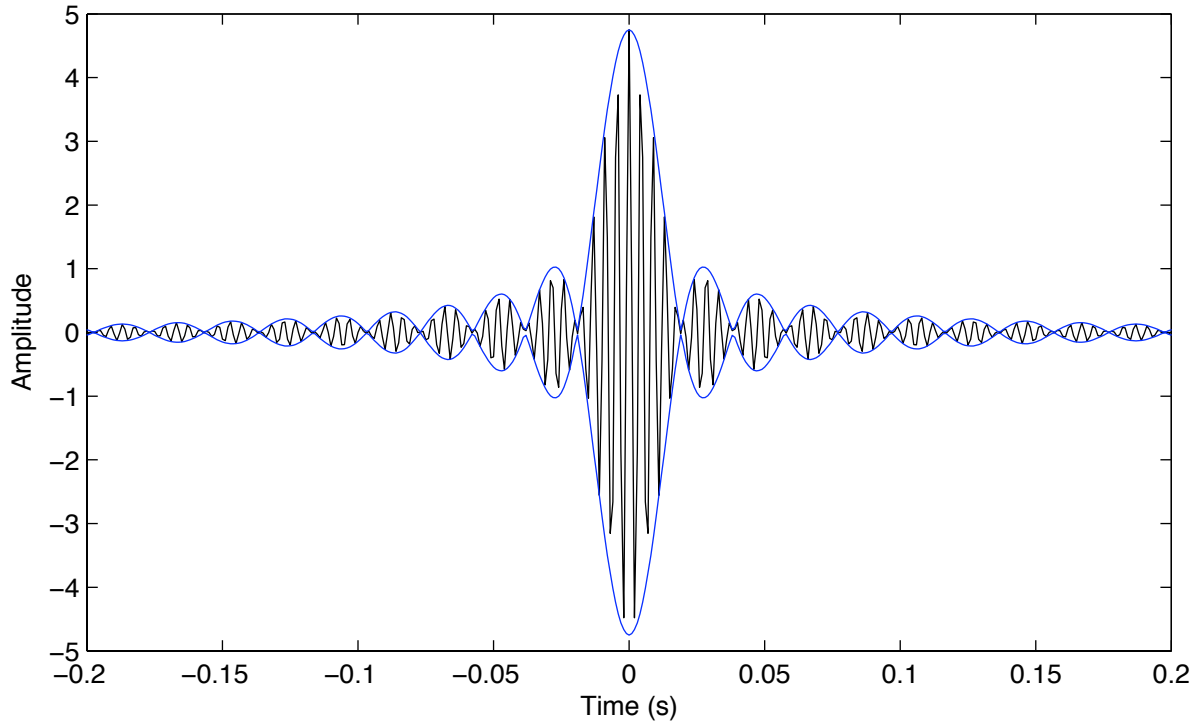
$$\tilde{s}(t) = s(t) - is^\dagger(t)$$

For a chirp signal it is often convenient to write the complex baseband signal in amplitude/phase format

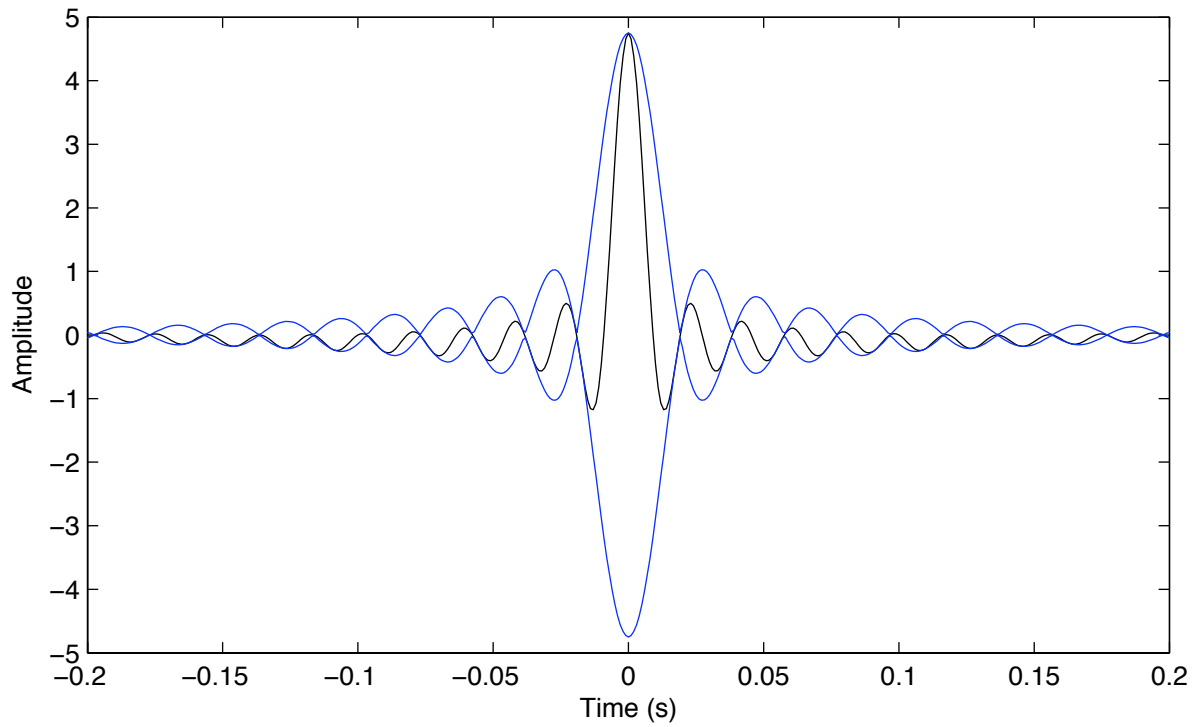
$$\tilde{s}(t) = w(t)e^{i\phi(t)}$$

The LFM & HFM signals considered previously can be regarded in this way, as baseband signals that have been modulated by a carrier frequency. One result that comes directly from this representation is that the complex envelope is invariant under modulation

$$|s(t)| = |\tilde{s}(t)| |e^{i2\pi f_c t}| = |\tilde{s}(t)|$$



Matched filtered response of LFM chirp, with envelope.



Matched filtered response of the 200-255 Hz LFM signal demodulated using 200 Hz. The MF response is now lower frequency and narrower but the envelope is the same.

Demodulation

In a sonar or chirp system, the received signal is often demodulated in the analog electronics using the same oscillator used to generate the carrier frequency for the transmitted signal.

The modulated signal can be written explicitly as

$$s(t) = s_I(t)\cos\omega_c t + s_Q(t)\sin\omega_c t$$

On being received by the sonar/radar the incoming signal is sent along two paths, beat against in phase and quadrature components of the carrier, that is, multiplied by $\sin\omega_c t$ and $\cos\omega_c t$, and then low pass filtered. For example the cosine multiplication yields

$$\frac{1}{2}[s_I(t)(1 + \cos 2\omega_c t) + s_Q(t)\sin 2\omega_c t]$$

Low pass filtering recovers $1/2 s_I(t)$, similarly the sine path recovers the quadrature component of the signal, in combination the two paths effectively recover the complex baseband signal only at half amplitude.

If the low pass filter was narrower than the signal then the demodulated output is convolved with the impulse response of the filter. If the filter is an ideal low pass filter with cutoff frequency $W/2$ then the resulting complex signal is actually

$$\hat{s}(t) = \frac{1}{2}\tilde{s}(t) * W\text{sinc}(\pi Wt)$$

One advantage of the demodulation is that by shifting to lower frequencies the sample rate can be reduced. The 200-255 Hz LFM signal would require a sample rate above 510 Hz to avoid aliasing (allowing for some spectral leakage), and would likely be records with 1 ms samples. After demodulation at, say, 225 Hz, the complex signal could be sampled at ~ 60 Hz, to avoid aliasing.

It would, of course, also be possible to effectively demodulate the trace simply by sampling at a slower rate and taking advantage of aliasing to accomplish the downshift in frequency. However, the sampling rate would need to be carefully chosen so that the resultant aliased frequencies were either all positive or all negative, otherwise the bandwidth of the (real) signal will be reduced and some information lost.